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SOME APPLICATIONS OF FRACTIONAL EQUATIONS

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Abstract

We present two observations related to the application of linear (LFE) and nonlinear fractional equations (NFE). First, we give the comparison and estimates of the role of the fractional derivative term to the normal diffusion term in a LFE. The transition of the solution from normal to anomalous transport is demonstrated and the dominant role of the power tails in the long time asymptotics is shown. Second, wave propagation or kinetics in a nonlinear media with fractal properties is considered. A corresponding fractional generalization of the Ginzburg-Landau and nonlinear Schrödinger equations is proposed.

1 Introduction

We call a fractional equation (FE) a differential equation that contains fractional derivatives or integrals. The awareness of the importance of this type of equation has grown continually in the last decade. Numerous applications have become apparent: wave propagation in a complex or porous media [1]-[3]; random walks with a memory and flights [4]-[6]; kinetic theories of systems with chaotic [7]-[9] and pseudochaotic [10] dynamics; and others (see [9] and collections of papers in [11]). This new type of problem has increased rapidly in areas in which the fractal features of a process or the medium impose the necessity of using non-traditional tools in "regular" smooth physical equations. Exploitation of the fractional calculus for FE provides not only new types of mathematical constructions, but also new physical features of the described phenomenon [4],[5]. While the linear FE has attracted fairly broad research activity, the study of nonlinear FE is at its very beginning [11]-[14].

This paper consists of two parts on the FE properties. The first part is related to a linear FE (LFE) with the second (diffusional) derivative and the fractional derivative of order α. A goal of this part is to present a solution to the FE and compare roles of different parts of the FE. The second part is related to a nonlinear FE (NLFE) with cubic nonlinearity. We discuss some features of these equations, and possible applications. For a broad discussion of the fractal features of different media where waves propagate, see the elegant review [3]). In particular, a turbulent media of this type may be considered. We will show that such media lead to NLFEs in a natural way after simple change of the dispersion law.

2 The interaction of normalized and anomalous transport

The interest in and relevance of kinetic equations with fractional derivatives is a natural consequence of the realization of the importance of non-Gaussian properties of the statistics of many dynamical systems. There is already a substantial literature studying such equations in one or more space dimensions. In many cases of physical interest it would be reasonable to assume that both Gaussian and anomalous processes would play a role. One typically associates anomalous processes with algebraically decreasing tails of a probability distribution function (PDF), while the bulk of the PDF is expected to be mostly Gaussian in chracter. In this note we explore the interaction of Gaussian and anomalous dynamics by means of a simple model kinetic equation in one space dimension for the PDF. More space dimensions would greatly complicate the analysis, but could readily be carried out.

We consider the kinetic equation with fractional derivatives

$$\frac{\partial P(x,t)}{\partial t'} = \frac{\partial^2 P}{\partial x'^2} + \epsilon \frac{\partial^\alpha P}{\partial |x'|^\alpha} \ , \qquad 1 < \alpha < 2 \tag{1}$$

where ϵ is a constant, presumed small, but not necessarily so. The Riesz derivatives $\frac{\partial^\alpha P}{\partial |x'|^\alpha}$ is easily defined for our purposes in Fourier transformed space, so that the Fourier transform of $\frac{\partial^\alpha P}{\partial |x'|^\alpha}$ is $-|k|^\alpha \tilde{P}(k,t)$ where $\tilde{P}(k,t)$ is the Fourier transform of $P(x',t)$. We have not taken a fractional derivative with respect to time. We could do so, but this would be yet another complication in the analysis. We see that the Fourier transform of the right hand side of (1) is $-(k^2 + \epsilon|k|^\alpha)\tilde{P}(,t)$. Thus, for large wavenumber and short wavelength the system exhibits normal, Gaussian transport, while for small wavenumber and large wavelength, the system is anomalous. The two phenomena are equal for wavenumber k_T

$$k_T = \epsilon^{\frac{1}{2-\alpha}}. \tag{2}$$

Thus (1) provides a simple model which mixes both normal and anomalous transport.

It is convenient to change the independent variables from x', t' to x, t with the definitions

$$x = \epsilon^{\frac{1}{2-\alpha}} x' \tag{3}$$

$$t = \epsilon^{\frac{2}{2-\alpha}} t' \tag{4}$$

so that

$$\frac{\partial P(x,t)}{\partial t} = \frac{\partial^2 P}{\partial x^2} + \frac{\partial^\alpha P}{\partial |x|^\alpha} \ . \tag{5}$$

The coordinate (x', t') are scaled to the Gaussian transport scale, while (x, t) are of a scale in which Gaussian and anomalous transport are comparable. We may easily represent a solution of (5) which also satisfies

$$P(x, 0) = \delta(x) \tag{6}$$

and as a consequence

$$\int_{-\infty}^{\infty} P(x,t) dx = 1 \ , \tag{7}$$

as a Fourier integral:

$$P(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \exp[-t(k^2 + |k|^\alpha) - ikx]. \tag{8}$$

We compare (8) with the solution of the equation with only anomalous transport

$$\frac{\partial Q(x,t)}{\partial t} = \frac{\partial^\alpha Q(x,t)}{\partial |x|^\alpha}, \tag{9}$$

where Q also satisfies (6) and (7), so that

$$Q(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \exp(-t|k|^\alpha - ikx). \tag{10}$$

The properties of $Q(x,t)$ have been well-explored in the literature. We need such results for comparison with the properties of $P(x,t)$.

We develop numerous series expansions of (8) and (10), some convergent and some asymptotic. When we know the behavior of $P(x,t)$ and $Q(x,t)$ for $|x|$ large, we shall also evaluate one spatial moment of $P(x,t)$ and $Q(x,t)$. With this information in hand we can then provide a description of the interaction of normal and anomalous transport. We start with (8), which we rewrite as

$$P(x,t) = \frac{\mathrm{Re}}{\pi}\int_0^\infty dk \exp[-t(k^2+k^\alpha) - ikx] \tag{11}$$

We observe that

$$e^{-tk^\alpha} = \sum_{n-0}^{N}(-1)^n t^n \frac{k^{\alpha n}}{n!} + R_{N+1}(k,t), \tag{12}$$

where

$$|R_{N+1}(k,t)| \le t^{N+1}\frac{k^{\alpha(N+1)}}{(N+1)!}. \tag{13}$$

Since

$$\left|\int_0^\infty dk \exp(-tk^2 - ikx)\right| \le \frac{t^{(N+1)}}{(N+1)!}\int_0^\infty dk\ k^{\alpha(N+1)} e^{-tk^2}$$

it is easy to conclude that the integral of the error tends to zero uniformly for all x and $0 < \eta \le t \le T$ as $N \to \infty$, so that

$$P(x,t) = \frac{\mathrm{Re}}{\pi}\sum_{n=0}^{\infty}(-1)^n \frac{t^n}{n!}\int_0^\infty dk\ k^{\alpha n}\exp(-tk^2 - ikx), \tag{14}$$

or

$$P(x,t) = \frac{1}{2\sqrt{\pi t}} e^{-\frac{x^2}{4t}} + \mathrm{Re}\frac{e^{-x^2/8t}}{\pi\sqrt{t}}\sum_{n=1}^{\infty}(-1)^n \frac{t^{n(1-\frac{\alpha}{2})}\Gamma(\alpha n+1)}{2^{\frac{\alpha n+1}{2}}\Gamma(n+1)} D_{-\alpha n-1}\left(\frac{ix}{\sqrt{2t}}\right), \tag{15}$$

where $D_\nu(z)$ is the Weber function of order ν. Again (15) converges uniformly for all x in $0 < \eta \le t \le T$. We see that the first term in (15) exhibits Gaussian diffusion and also has unit integral. Hence, the remaining terms, the anomalous transport effects, have zero mean. For $|x|$ large and n fixed we may employ the asymptotic expansion of the Weber function of large argument, or directly from (15) we obtain the asymptotic expansion for large $|x|$:

$$\frac{1}{\pi}\mathrm{Re}(-1)^n \frac{t^n}{n!}\int_0^\infty dk\ k^{\alpha n}\exp(-tk^2 - ikx)$$

$$= \frac{(-1)^{n+1}}{\pi}\ \frac{t^b}{n!|x|^{\alpha n+1}}\sin\left(\frac{\alpha\pi n}{2}\right)\left\{\sum_{m=0}^{M}\left(\frac{t}{x^2}\right)^m \frac{\Gamma(\alpha n+2m+1)}{m!} + O\left(\frac{t}{x^2}\right)^{M+1}\right\} \tag{16}$$

The series (16) is clearly asymptotic and shows the power tail of the PDF, with the leading term from $n=1$ being

$$\frac{(t/|x|^\alpha)}{\pi|x|}\Gamma(\alpha+1)\sin\frac{\pi\alpha}{2}\ . \tag{17}$$

We may obtain another convergent expansion by use of the power series

$$e^{-ikx} = \sum_{n=0}^{N} (-i)^n \frac{k^n x^n}{n!} + R_{N+1}(k,x), \tag{18}$$

where

$$|R_{N+1}(k,x)| \leq k^{N+1} \frac{|x|^{N+1}}{(N+1)!} \tag{19}$$

and thus

$$P(x,t) = \sum_{n=0}^{\infty} (-1)^n x^{2n} C_n(t), \tag{20}$$

where

$$C_n(t) = \frac{1}{(2n)!\pi} \int_0^\infty dk\ k^{2N} \exp -t(k^2 + k^\alpha). \tag{21}$$

Although $C_n(t)$ does ot appear to be one of the usual special functions it is easy to obtain an asymptotic expansion for t large and convergent expansion for t small. In particular for t small but bounded away from zero:

$$C_n(t) = \frac{t^{-n-1/2}}{(2n)!2\pi} \sum_{n=0}^{\infty} \frac{(-1)^m}{m!} t^{m(1-\alpha/2)} \Gamma\left(n + \frac{\alpha m}{2} + \frac{1}{2}\right) \tag{22}$$

and for t large

$$C_n(t) = \frac{t^{-(2n+1)/\alpha}}{(2n)!2\pi} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} t^{m(1-\frac{2}{\alpha})} \Gamma\left(\frac{2(n+m)+1}{2}\right). \tag{23}$$

We see from (14) and (17) that for $|x|$ large $P(x,t) \sim (\text{const.})|x|^{-\alpha-1}$. Thus, moments of order $\geq \alpha$ do not exist. We examine one, simple relevant moment

$$M = \int_{-\infty}^{\infty} |x| dx\ P(x,t) \tag{24}$$

and it follows easily that

$$M = \frac{2}{\pi} \int_0^\infty dx\ \mathrm{Re} \int_0^\infty dk\ \ \exp[-t(k^2 + k^\alpha)] \frac{d}{dk} \sin kx. \tag{25}$$

After integration by parts we obtain

$$M = \frac{2t}{\pi} \int_0^\infty dx\ \mathrm{Im} \int_0^\infty dk (2k + \alpha k^{\alpha-1}) \exp[-t(k^2 + k^\alpha)] e^{ikx} \tag{26}$$

If we move the path of x integration into the upper half-plane we find

$$M = \frac{2t}{\pi} \int_0^\infty dk (2 + \alpha k^{\alpha-2}) \exp -t(k^2 + k^\alpha) \tag{27}$$

With the same type of expansion as before we find an expansion convergent in $0 < \eta \leq t \leq T$

$$M = \frac{t^{1/2}}{\pi} \sum_{m=0}^{\infty} (-1)^{m+1} \frac{t^{m(1-\alpha/2)}}{m!} \Gamma\left(\frac{m\alpha - 1}{2}\right) \tag{28}$$

The leading order term $m = 0$ is just $2(t/\pi)^{1/2}$, exactly what one would obtain from a Gaussian. The remaining terms are the corrections from anomalous transport. For t large we have the asymptotic expansion

$$M = \frac{2t^{1/\alpha}}{\pi} \sum_{m=0}^{\infty} (-1)^m t^{-m(\frac{2}{\alpha}-1)} \frac{(-\frac{1}{\alpha})\Gamma(\frac{2m-1}{\alpha})}{m!} \tag{29}$$

We now turn to $Q(x,t)$ so that we may assess the significance of the preceding results. We start from (10) and we find easily the convergent expansion

$$Q = \frac{1}{\pi\alpha} \sum_{n=0}^{\infty} (-1)^n t^{-1/2} [x/t^{1/\alpha}]^{2n} \frac{\Gamma(\frac{2n+1}{\alpha})}{\Gamma(2n+1)} \tag{30}$$

while for $|x|$ large there is the asymptotic expansion

$$Q \sim \frac{1}{\pi|x|} \sum_{n=1}^{\infty} (-1)^n (t/|x|^\alpha)^n \frac{\Gamma(n\alpha+1)}{\Gamma(n+1)} \sin\frac{n\pi\alpha}{2}. \tag{31}$$

Finally, we define M_Q as

$$M_Q = \int_{-\infty}^{\infty} Q(x,t)|x|dx \tag{32}$$

so that

$$M_Q = \frac{2}{\pi} \int_0^\infty x dx \; \mathrm{Re} \int_0^\infty dk \exp(-k^\alpha t) \frac{d}{dk} \sin kx. \tag{33}$$

We follow the same procedure as for M and we obtain

$$M_Q = \frac{2}{\pi} t^{1/\alpha} \Gamma(1 - 1/\alpha). \tag{34}$$

We are now prepared to compare results and we start with the simplest comparison, M and M_Q. The result (34) for ν is valid for all times, however $M(t)$ is very different from (34) for $t \sim 1$. For t large (29) shows

$$M \sim \frac{2}{\pi} \left[t^{1/\alpha} \Gamma\left(1 - \frac{1}{\alpha}\right) + t^{1-1/\alpha} \Gamma\left(1 + \frac{1}{\alpha}\right) + O(t^{2-\frac{3}{\alpha}}) \right]. \tag{35}$$

Thus, in leading order and for $1 < \alpha < 2$, $M \sim M_Q$. However, the difference between M and M_Q is not small unless $t^{1/\alpha} \gg t^{1-\frac{1}{\alpha}}$. For α not far from one, this relation holds, but as α approaches 2, one requires increasing larger values of t in order that $t^{1/\alpha}$ dominate $t^{1-\frac{1}{\alpha}}$. It is clear that the expansion (29) fails at $\alpha = 2$, and must then be very poor for small values of $2 - \alpha$. An examination of this moment indicates that for t sufficiently large the anomalous transport is the limiting form for the case with both Gaussian and anomalous transport, however there may be significant corrections.

When we compare $P(x,t)$ and $Q(x,t)$ the situation is somewhat more complex. For t and $x \lesssim 1$, so that both Gaussian and normal transport may occur we may compare (15),(16) with (30),(31). When $|x| \sim 1$, P and Q are substantially different; however for $t \sim 1$, $|x| > 1$, one may use (16), and then the leading order terms in (16) match (3) although there are nontrivial corrections of order

t/x^2. When $t > 1$ but $|x| \sim 1$, we may compare (21) and (23) with (30). Again the leading order terms agree but there are nontrivial corrections of order $t^{-(\frac{2}{\alpha}-1)}$. Finally, when both x and t are large, we must decide whether $|x|^\alpha/t$ is large or not. If $|x|^\alpha/t \sim 1$ after a little effort we see that we may use (20) and (23) for $P(x,t)$, which again matches $Q(x,t)$ in (30), but with corrections. Finally if $|x|$ and t are both large, as is $|x|^\alpha/t$, then we may conclude that (14) and (16) apply for $P(x,t)$ which approximates $Q(x,t)$ as given by (31). We see that broadly the anomalous transport finally dominates the Gaussian, or normal, transport, although it may take a long time. From another point of view, unless ϵ is extremely small, the anomalous transport description is generally more relevant than the normal transport, although there may be significant corrections to the anomalous transport results.

3 Fractional generalization of Ginzburg-Landau equation and nonlinear Schrödinger equation

Let us recall the appearance of the nonlinear parabolic equation (see for example [15]). The simplest way is to consider a symmetric dispersion law $\omega = \omega(k)$ for wave propagation in some media, and to represent the wave vector $\mathbf{k}$ in the form

$$\mathbf{k} = \mathbf{k}_0 + \boldsymbol{\kappa} = \mathbf{k}_0 + \boldsymbol{\kappa}_\parallel + \boldsymbol{\kappa}_\perp \tag{36}$$

where $\mathbf{k}_0$ is the unperturbed wave vector and subscripts $(\parallel, \perp)$ are taken respectively to the direction $\mathbf{k}_0$. Considering $\kappa \ll k_0$ we have

$$\begin{aligned} \omega(k) &= \omega(|\mathbf{k}_0 + \boldsymbol{\kappa}|) \approx \omega(k_0) \\ &+ \ c(|\mathbf{k}_0 + \boldsymbol{\kappa}| - k_0) \approx \omega(k_0) + c\kappa_\parallel + \frac{c}{2k_0}\kappa_\perp^2 \end{aligned} \tag{37}$$

where $c = \partial\omega/\partial k_0$. The expression (37) corresponds to the linear parabolic equation

$$-i\frac{\partial Z}{\partial t} = ic\frac{\partial Z}{\partial x} + \frac{c}{2k_0}\Delta Z \tag{38}$$

with respect to a field $Z = Z(x, \mathbf{r}, t)$, x along the $\mathbf{k}_0$, and the operator correspondence:

$$\nu \equiv \omega(k) - \omega(k_0) = i\frac{\partial}{\partial t}, \quad \kappa_\parallel = -i\frac{\partial}{\partial x}, \quad \boldsymbol{\kappa}_\perp = -i\frac{\partial}{\partial \mathbf{r}}, \quad \mathbf{r} = (y, z) \tag{39}$$

A generalization to a nonlinear case can be carried out in analogy with (37) through a nonlinear dispersion law depending on the wave amplitude:

$$\omega = \omega(k, |Z|^2) \approx \omega(k, 0) + g|Z|^2$$

$$= \omega(|\mathbf{k}_0 + \boldsymbol{\kappa}|, 0) + g|Z|^2 \tag{40}$$

with some constant $g = \partial\omega(k, |Z|^2)/\partial|Z|^2$ at $|Z|^2 = 0$. In analogy with (38), the nonlinear parabolic equation takes the form

$$-i\frac{\partial Z}{\partial t} = ic\frac{\partial Z}{\partial x} + \frac{c}{2k_0}\Delta Z - g|Z|^2 Z \tag{41}$$

This equation is also known as the nonlinear Schrödinger equation in which any of the coefficients may be complex. Indeed, for a traveling wave $Z = Z(x - ct, \mathbf{r})$ we have

$$-ic\frac{\partial Z}{\partial \xi} = \frac{c}{2k_0}\Delta Z - g|Z|^2 \tag{42}$$

with $\xi = x - ct$.

Wave propagation in a media with fractal propeties can be easily generalized from the first step of writing dispersion law (37). Namely, one can replace (37) and (40) by the following:

$$\nu = \omega(k, |Z|^2) - \omega(k_0, 0) = c\kappa_\parallel + c_\alpha(\boldsymbol{\kappa}_\perp^2)^{\alpha/2} + g|Z|^2 \tag{43}$$

with a fractional value of $1 < \alpha < 2$ and new constant c_α. This replacement does not affect the nonlinear term which appears for $\kappa_\parallel = \kappa_\perp = 0$ (see (40)).

Using the connection between the differentiation operator and its Fourier transform

$$(-\Delta)^{\alpha/2} \leftrightarrow (\boldsymbol{\kappa}_\perp^2)^{\alpha/2} \tag{44}$$

we obtain the equation corresponding to (43):

$$-i\frac{\partial Z}{\partial t} = ic\frac{\partial Z}{\partial x} - \frac{c}{2k_0}(-\Delta)^{\alpha/2} Z - g|Z|^2 Z \tag{45}$$

or for travelling waves as in (42)

$$ic\frac{\partial Z}{\partial \xi} = \frac{c}{2k_0}(-\Delta)^{\alpha/2} Z + g|Z|^2 Z \tag{46}$$

Let us comment on the physical structure of (46). The first term on the right-hand side is related to the wave propagation in a media with fractal properties. The fractal derivative may also appear as a result of ray chaos [16], or of a superdiffusive wave propagation (see also the discussion in [3],[9] and corresponding references). The second term on the right-hand side of (45),(46) corresponds to the wave interaction due to the nonlinear properties of the media. Thus, (46) describes fractional processes of self-focusing and related issues.

We may consider a one-dimensional simplification of (46), i.e.

$$c\frac{\partial Z}{\partial \xi} = a\frac{\partial^\alpha Z}{\partial |z|^\alpha} + g|Z|^2 Z \tag{47}$$

with some generalized constants c, a, g and then reduce (47) to the case of a propagating wave solution

$$Z = Z(z - c\xi) \equiv Z(\eta). \tag{48}$$

Then (47) takes the form of fractional generalization of the Ginzburg-Landau equation (FGL):

$$a\frac{d^\alpha Z}{d|\eta|^\alpha} + \frac{dZ}{d\eta} + gZ^3 = 0 \tag{49}$$

for real $Z(\eta)$. Now (49) differs from the fractional Burgers equation [13],[14] in the structure of the nonlinear term. Nevertheless, an analysis similar to [13],[14] may be performed to obtain some estimates on the solution.

It is well known that the nonlinear term in the equations of (42) type leads to a steepening of the solution and its singularity. The steepening process may be stopped by a diffusional or dispersional term, i.e. by a higher derivative term. A similar phenomenon may appear for the fractional nonlinear equations (47),(49). It has been shown in [13] that for the fractional Burgers equation there exists a critical value α_c such that solution is regular for all time if $\alpha > \alpha_c$.

Next, we consider the symmetric solution $Z(\eta) = Z(-\eta)$ of (48) and assume $Z \in I\!R^2$. Let us multiply (49) by Z and integrate it under the assumption that Z is sufficiently small at infinity. We find

$$\mathcal{E} \equiv \int_{-\infty}^{\infty} (aZZ^{(\alpha/2)} + gZ^4)d\eta = 0 \tag{50}$$

where we use the notation

$$\frac{d^\alpha Z}{d|\eta|^\alpha} \equiv Z^{(\alpha)} \tag{51}$$

The expression (50) gives a conservation law showing the relative importance of the nonlinear and dispersion terms. Considering $Z^{(\alpha)} \sim 1/\eta^\gamma$ we conclude that the term with derivative prevails for small η if

$$\alpha > 2\gamma. \tag{52}$$

For $2 > \alpha > 1$ the condition (52) shows that the solution is square integrable at $\eta \to 0$.

Finally we observe that (49) may describe a new type of fractional dynamic motion. This topic will be considered elsewhere.

4 Conclusion

In this brief paper we have made two remarks related to application of fractional equations in physics. The first remark is related to the competition between normal diffusion and diffusion induced by fractional derivatives, as occur in fractional kinetic theories. It is shown that for large times the

fractional derivative term dominates the solution and leads to power type tails. The second remark is related to a new class of equations in which fractional derivative terms are responsible for fractional dispersion. In this case the asymptotics should be defined by a competition between the fractional dispersion and nonlinear terms. We discuss the origin of the fractional Ginzburg-Landau equation and fractional nonlinear Schrödinger equation.

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